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ZEROS OF A^p FUNCTIONS AND RELATED CLASSES OF ANALYTIC FUNCTIONS

BY

ELIYAHU BELLER

ABSTRACT

The function $f(z)$, analytic in the unit disc, is in A^p if $\int\int_{|z|<1} |f(z)|^p dxdy < \infty$. A necessary condition on the moduli of the zeros of A^p functions is shown to be best possible. The function $f(z)$ belongs to B^p if $\int\int_{|z|<1} \log^+ |f(z)|)^p dxdy < \infty$. Let $\{z_n\}$ be the zero set of a B^p function. A necessary condition on $|z_n|$ is obtained, which, in particular, implies that $\Sigma(1-|z_n|)^{1+(1/p)+\varepsilon} < \infty$ for all $\varepsilon > 0$ $(p \geqq 1)$. A condition on the Taylor coefficients of f is obtained, which is sufficient for inclusion of f in B^p. This in turn shows that the necessary condition on $|z_n|$ is essentially the best possible. Another consequence is that, for $q \geqq 1$, $p < q$, there exists a B^p zero set which is not a B^q zero set.

1. Zeros of A^p functions

A function $f(z)$, analytic in the unit disc, is said to belong to the space A^p, $0 < p < \infty$, if

$$\|f\|_p^p = \frac{1}{\pi} \int\int_{|z|<1} |f(z)|^p dxdy < \infty.$$

Let $f \in A^p$, and let $\{z_n\}$ be the sequence of its zeros, repeated according to their multiplicity, such that $|z_1| \leqq |z_2| \leqq |z_3| \leqq \cdots$ (the "ordered zeros" of f). What can one say about the rate of convergence of $|z_n|$ to the unit circumference?

As pointed out by B. Epstein [1], since $|f(z)|^p$ is subharmonic in $|z| < 1$, we have, for $|t| < 1$,

$$|f(t)|^p \leqq \frac{1}{\pi(1-|t|)^2} \int\int_{|z-t|<1-|t|} |f(z)|^p dxdy < \frac{\|f\|_p^p}{(1-|t|)^2},$$

so that

$$(1) \qquad |f(t)| < \frac{A}{(1-|t|)^{2/p}} \qquad (|t| < 1),$$

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with $A = \|f\|_p$. H. S. Shapiro and A. L. Shields [4] proved that, if f is analytic in the unit disc and satisfies (1), then

$$n(r) = O\left(\frac{1}{1-r}\log\frac{1}{1-r}\right),$$

where $n(r)$ denotes the number of zeros of f in $|z| < r$. This is equivalent to saying that

$$1 - |z_n| = O\left(\frac{\log n}{n}\right). \tag{2}$$

Thus, in particular, $\Sigma(1 - |z_n|)^{1+\varepsilon} < \infty$ for every $\varepsilon > 0$, so that $\{z_n\}$ "almost" satisfies the Blaschke condition $\Sigma(1 - |z_n|) < \infty$, which holds for H^p spaces.

If one follows the estimates of Shapiro and Shields, it is not hard to see that the constant behind the big "O" in (2) is $4/p$. C. Horowitz [3], who obtained many interesting results concerning A^p zero sets, proved a theorem which brings the constant down to $1/p$. Specifically, he proved that if $f \in A^p$, $(0 < p < \infty)$, and $f(0) \neq 0$, then

$$\prod_{k=1}^{n} |z_k|^{-1} \leqq c n^{1/p}, \tag{3}$$

where $\{z_n\}$ are the ordered zeros of f, and c depends only on f and p. From (3) it follows that

$$\sum_{k=1}^{n} (1 - |z_k|) < \sum_{k=1}^{n} -\log|z_k| \leqq \log c + \frac{1}{p}\log n, \tag{4}$$

and so

$$\limsup_{n\to\infty} (\log n)^{-1} \sum_{k=1}^{n} (1 - |z_k|) \leqq 1/p. \tag{5}$$

Since $\{|z_k|\}$ is nondecreasing, we have $\Sigma_{k=1}^{n}(1 - |z_k|) \geqq n(1 - |z_n|)$, and combining this with (4), we conclude that[†]

$$\limsup_{n\to\infty} \frac{1 - |z_n|}{n^{-1}\log n} \leqq \frac{1}{p}. \tag{6}$$

From Horowitz's work [3], it follows that the constant $1/p$ in (5) is sharp, but this in itself does not imply the sharpness of (6). (It does not even exclude the possibility that the lim sup in (6) is always zero.) We will now show that the constant $1/p$ in (6) is indeed the best possible.

[†] The lim sup of $(1 - |z_n|)/(n^{-1}\log n)$ was first studied by Epstein [1].

THEOREM 1. *For each* p $(0 < p < \infty)$, *and for each* $\varepsilon > 0$, *there exists an* $f \in A^p$ *such that*

$$\limsup_{n \to \infty} \frac{1 - |z_n|}{n^{-1} \log n} > \frac{1}{p(1+\varepsilon)}, \tag{7}$$

where $\{z_n\}$ *are the ordered zeros of* f.

PROOF. Let $\varepsilon > 0$. Choose a natural $N \geqq 2$ such that $(N-1)^{-1} < \varepsilon$. Set

$$f(z) = \prod_{k=0}^{\infty} [1 + b_k z^{(n_k - n_{k-1})}], \tag{8}$$

where $n_k = 2^{N^k}$ $(k = 0, 1, 2, \cdots,)$, $n_{-1} = 0$, and $b_k = n_k^{\{p(1+\varepsilon)\}^{-1}}$. Since the radius of convergence of $\sum_{k=0}^{\infty} b_k z^{(n_k - n_{k-1})}$ is 1, it follows that f is analytic in $|z| < 1$, and its zeros are precisely the zeros of the factors of the right side of (8). Clearly, for $n_{k-1} < n \leqq n_k$, we have

$$|z_n| = b_k^{-(n_k - n_{k-1})^{-1}} = n_k^{-\{p(1+\varepsilon)(n_k - n_{k-1})\}^{-1}}.$$

In particular,

$$1 - |z_{n_k}| = 1 - \exp\left(\frac{-\log n_k}{p(1+\varepsilon)(n_k - n_{k-1})}\right)$$

$$> \frac{\log n_k}{p(1+\varepsilon)(n_k - n_{k-1})} - \frac{\log^2 n_k}{2p^2(1+\varepsilon)^2(n_k - n_{k-1})^2} > \frac{\log n_k}{p(1+\varepsilon)n_k}$$

for sufficiently large k, which proves (7). Thus, it remains to be shown that our f belongs to A^p.

We will use the following theorem of Horowitz [3]:

THEOREM H. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be analytic in the unit disc and let* $S_n^{(q)} = \sum_{k=0}^{n} |a_k|^q$ $(0 < q < \infty)$.

(i) *For* $0 < p \leqq 2$, *if* $S_n^{(2)} = O(n^a)$ *for some* $a < 2/p$, *then* $f \in A^p$.

(ii) *For* $2 \leqq p < \infty$, *let* $q = p/(p-1)$. *If* $S_n^{(q)} = O(n^a)$ *for some* $a < q - 1$, *then* $f \in A^p$.

We first need the following

LEMMA 1. *For the* f *defined in* (8), *we have*

$$S_{n_m}^{(q)} = \prod_{k=0}^{m} (1 + b_k^q) \quad (0 < q < \infty;\ m = 0, 1, 2, \cdots).$$

PROOF OF LEMMA 1. We use induction on m. It is true for $m = 0$; assume that $S_{n_m}^{(q)} = \prod_{k=0}^{m} (1 + b_k^q)$. Set

$$f_m(z) = \prod_{k=0}^{m} [1 + b_k z^{(n_k - n_{k-1})}].$$

Note that $f_m(z) = \sum_{k=0}^{n_m} a_k z^k$, since $\deg(f_m) = n_m < n_{m+1} - n_m$. Thus,

$$f_{m+1}(z) = [1 + b_{m+1} z^{(n_{m+1} - n_m)}] f_m(z) = \sum_{k=0}^{n_m} a_k z^k + \sum_{k=0}^{n_m} a_k b_{m+1} z^{(k + n_{m+1} - n_m)},$$

so that

$$S^{(q)}_{n_{m+1}} = \sum_{k=0}^{n_m} |a_k|^q (1 + b^q_{m+1}) = \prod_{k=0}^{m+1} (1 + b^q_k). \qquad \blacksquare$$

We now proceed to prove that $f \in A^p$. Let us first consider the case $0 < p \leqq 2$. Using Lemma 1, we have

$$S^{(2)}_{n_m} = \prod_{k=0}^{m} (1 + b^2_k) < \prod_{k=0}^{\infty} (1 + b_k^{-2}) \prod_{k=0}^{m} b^2_k = K \prod_{k=0}^{m} b^2_k = K \prod_{k=0}^{m} 2^{2N^k/(p(1+\varepsilon))}$$

$$= K 2^{2(N^{m+1}-1)/\{p(1+\varepsilon)(N-1)\}} < K n_m^{2(1+(N-1)^{-1})/\{p(1+\varepsilon)\}}.$$

Thus, for $m = 0, 1, 2, \cdots$, we have $S^{(2)}_{n_m} < K n^a_m$, where $a < 2/p$. Let us now look at the n's between n_{m-1} and n_m. For $n_{m-1} < n < n_m - n_{m-1}$, we have

$$S^{(2)}_n = S^{(2)}_{n_{m-1}} < K n^a_{m-1} < K n^a,$$

while for $n_m - n_{m-1} \leqq n \leqq n_m$, we have

$$S^{(2)}_n \leqq S^{(2)}_{n_m} < K n^a_m \leqq K \{n_m/(n_m - n_{m-1})\}^a \cdot n^a < K 2^a n^a.$$

Thus, $S^{(2)}_n = O(n^a)$, and by Theorem H(i), $f \in A^p$.

For the case $2 < p < \infty$, with $q = p/(p-1)$, it can be shown, by almost identical reasoning, that $S^{(q)}_n = O(n^a)$, where $a < q - 1$, and so by Theorem H(ii) we again have $f \in A^p$. This concludes the proof of Theorem 1.

REMARK. Horowitz [2] generalized (3) by proving that if $f \in A^p$, and $\{z_k\}$ are its (ordered) zeros in the sector $\theta_1 \leqq \arg z \leqq \theta_2$, then

$$\prod_{k=1}^{n} |z_n|^{-1} \leqq c n^{\beta/p}, \quad \text{where} \quad \beta = (\theta_2 - \theta_1)/(2\pi).$$

The functions (8) constructed above obviously prove that this result for sectors, together with the resulting analog of (6), is the best possible.

2. B^p spaces

For each A^p space, $0 < p < \infty$, (6) gives us upper bounds for $1 - |z_n|$ which, as we have seen, are the best possible. This leads one to inquire about the situation for the class of functions f, analytic in $|z|<1$, for which the integral $\iint_{|z|<1} \log^+|f(z)|\,dxdy$ is finite (corresponding to the Nevanlinna class N in H^p theory). We can be a bit more general; namely, for each p, $0<p<\infty$, let B^p be the set of functions $f(z)$, analytic in the unit disc, for which

$$\iint_{|z|<1} (\log^+|f(z)|)^p dxdy < \infty.$$

Aside from the obvious inclusion $B^q \subset B^p$ for $p<q$, let us note that B^p $(0<p<\infty)$ is actually an algebra under pointwise addition and multiplication. Indeed, the inequality

$$\log^+|f+g| \leqq \log^+|f| + \log^+|g| + \log 2$$

shows that B^p is closed under addition, while closure under multiplication follows from

$$\log^+|fg| \leqq \log^+|f| + \log^+|g|.$$

This implies that the union of two B^p zero sets is a B^p zero set, which contrasts with the fact that the union of two A^p zero sets is an $A^{p/2}$ zero set, but not necessarily an A^q zero set for $q>p/2$ (see Horowitz [3]). In particular, this means that we cannot expect to have such a "tight" hierarchy of upper bounds for $1-|z_n|$, as exhibited by (6), which differ only by a multiplicative parameter depending on p.

We now state the analog of (6) for B^p functions:

THEOREM 2. *Suppose $f \in B^p$, $1 \leqq p < \infty$, and let $\{z_n\}$ be its ordered zeros. Then*

$$1-|z_n| = o(n^{-p/(p+1)}).$$

COROLLARY. *If $f \in B^p$, $1 \leqq p < \infty$, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} (1-|z_n|)^{1+(1/p)+\varepsilon} < \infty.$$

PROOF OF THEOREM 2. Set

$$M_r = (2\pi)^{-1}\int_0^{2\pi} (\log^+|f(re^{i\theta})|)^p d\theta, \qquad (0 \leqq r < 1).$$

Given $\varepsilon > 0$, let

$$A = \{r \mid M_r > \varepsilon/(1-r)\}.$$

The fact that f belongs to B^p implies that M_r is integrable on $0 \leqq r < 1$. Roughly speaking, this tells us that A becomes "rarer" as $r \to 1$. To make this idea precise, let $I_n = [1 - 2^{-n+1}, 1 - 2^{-n})$, so that $[0,1) = \bigcup_{n=1}^{\infty} I_n$. Then

$$\lim_{n\to\infty} \frac{\mu(A \cap I_n)}{\mu(I_n)} = \lim_{n\to\infty} 2^n \mu(A \cap I_n) = 0. \tag{9}$$

Indeed, assume the contrary of (9), i.e., assume that there exists a subsequence $\{n_k\}$ and a number $c > 0$ such that

$$\mu(A \cap I_{n_k}) \geqq c\, 2^{-n_k} \qquad (k = 1, 2, 3, \cdots).$$

Then

$$\int_0^1 M_r dr \geqq \int_A \{\varepsilon/(1-r)\}\, dr \geqq \sum_{k=1}^{\infty} \int_{A \cap I_{n_k}} \{\varepsilon/(1-r)\}\, dr$$

$$> \sum_{k=1}^{\infty} 2^{(n_k - 1)} \mu(A \cap I_{n_k}) \geqq \frac{\varepsilon}{2} \sum_{k=1}^{\infty} 2^{n_k} c\, 2^{-n_k} = \infty,$$

contradicting the integrability of M_r.

Now let K be such that $\mu(A \cap I_k) \leqq 3^{-1} 2^{-k}$ for $k \geqq K$. For each natural n, define $k(n)$ to be the index of the subinterval I_k in which $|z_n|^{\frac{1}{2}}$ lies: $|z_n|^{\frac{1}{2}} \in I_k$. For sufficiently large n, $k(n)$ will be greater than K. For such n, it is always possible to choose an $r_n \notin A$ such that

$$|z_n|^{\frac{1}{2}} \leqq r_n \leqq |z_n|^{\frac{1}{2}} + 2^{-1} 2^{-k}. \tag{10}$$

Assume for simplicity that $f(0) = 1$. For each r_n, Jensen's theorem yields

$$\sum_{|z_i| \leqq r_n} \log \frac{r_n}{|z_i|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(r_n e^{i\theta})|\, d\theta \leqq \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(r_n e^{i\theta})|\, d\theta$$

$$\leqq \left(\frac{1}{2\pi} \int_0^{2\pi} (\log^+ |f(r_n e^{i\theta})|)^p d\theta \right)^{1/p} \leqq \frac{\varepsilon^{1/p}}{(1 - r_n)^{1/p}}.$$

(The second inequality holds because $p \geqq 1$, while the last inequality follows from the fact that $r_n \notin A$.) Using (10), we have

$$(1 - r_n)^{-1/p} \leqq (1 - |z_n|^{\frac{1}{2}})^{-1/p} (1 - (1 - |z_n|^{\frac{1}{2}})^{-1} 2^{-k})^{-1/p}$$

$$\leqq 2^{1/p} (1 - |z_n|^{\frac{1}{2}})^{-1/p} \leqq 2^{2/p} (1 - |z_n|)^{-1/p}.$$

Putting everything together, we have

$$\sum_{|z_i| \leqq r_n} \log \frac{r_n}{|z_i|} < \frac{2^{2/p} \varepsilon^{1/p}}{(1 - |z_n|)^{1/p}}. \tag{11}$$

On the other hand,

$$\sum_{|z_i|\leq r_n} \log \frac{r_n}{|z_i|} \geqq \sum_{|z_i|\leq |z_n|^{1/2}} \log \frac{|z_n|^{\frac{1}{2}}}{|z_i|} \geqq \sum_{|z_i|\leq |z_n|} \log \frac{|z_n|^{\frac{1}{2}}}{|z_i|}$$

$$\geqq \sum_{|z_i|\leq |z_n|} \log |z_n|^{-\frac{1}{2}} \geqq -\tfrac{1}{2} n \log |z_n| \geqq \tfrac{1}{2} n (1 - |z_n|).$$

Combining this with (11), we conclude that, for sufficiently large n,

$$1 - |z_n| \leqq 2^{(p+2)/(p+1)} \varepsilon^{1/(p+1)} n^{-p/(p+1)}.$$

Thus,

$$\limsup_{n\to\infty} n^{p/(p+1)}(1 - |z_n|) \leqq c_p \varepsilon^{1/(p+1)}.$$

Since this is true for every $\varepsilon > 0$, we finally conclude that

$$1 - |z_n| = o(n^{-p/(p+1)}).$$

If $f(z)$ has a zero of order m at the origin, $(m = 0, 1, 2, \cdots)$, one needs only to divide by $f^{(m)}(0)z^m/m!$ before applying the above argument.

Our next task is to find conditions on the Taylor coefficients of f which insure that $f \in B^p$. With the aid of these conditions, we will be able to show that the exponent of n in Theorem 2 is the best possible. Another consequence will be that for $p < q$, $(q \geqq 1)$, there exists a B^p zero set which is not a B^q zero set. But first, we need the following asymptotic formula:

LEMMA 2. *Let $a > 0$, and let $\{c_n\}$ be defined by*

$$g(x) = \exp\{(1-x)^{-a}\} = \sum_{n=0}^{\infty} c_n x^n.$$

Then

$$c_n \sim K(a) n^{-(a+2)/(2a+2)} \exp(\{a^{-a/(a+1)} + a^{1/(a+1)}\} n^{a/(a+1)})$$

as $n \to \infty$, where $K(a) = (2\pi a(a+1))^{-1/2}$.

PROOF. By the Cauchy formula,

$$c_n = \frac{g^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_C z^{-n-1} g(z)dz = \frac{1}{2\pi i}\int_C \exp(\psi_n(z))dz,$$

where $\psi_n(z) = e^{-a\log(1-z)} - (n+1)\log z$, and C is any simple closed path around the origin which leaves the singularity at $z = 1$ in its exterior.

Let u_n be the unique real value of z at which $\psi'(z)$ vanishes, so that $au_n(1-u_n)^{-1} = n+1$. Thus,

$$u_n = 1 - a^{1/(a+1)} n^{-1/(a+1)} + O(n^{-2/(a+1)}). \tag{12}$$

Henceforth we will suppress the index n of u_n.

Let the path C be composed of the vertical line through u from $u-iR$ to $u+iR$, $(R>1)$, together with the left hand semicircle with center u and radius R, described counterclockwise. It is easy to see that as $R\to\infty$, the integral on the semicircle goes to zero. Thus, writing $au(1-u)^{-1}$ for $n+1$ in the integrand, we have

$$\begin{aligned}\frac{g^{(n)}(0)}{n!} &= \frac{1}{2\pi i}\int_{u-i\infty}^{u+i\infty} \exp(\psi_n(z))\,dz \\ &= \frac{1}{2\pi}\exp\{(1-u)^{-a} + (n+1)\log u\}\int_{-\infty}^{\infty} \exp(\phi_n(y))dy,\end{aligned} \tag{13}$$

where

$$\phi_n(y) = \exp\{-a\log(1-u-iy)\} - (1-u)^{-a} - au(1-u)^{-a-1}\log(1+iy/u).$$

We will now show that the integral on $(-\infty, -\frac{1}{2}(1-u)] \cup [\frac{1}{2}(1-u), \infty)$ is exponentially small as $n\to\infty$. Specifically, it is claimed that

$$\left\{\int_{-\infty}^{-1/2(1-u)} + \int_{1/2(1-u)}^{\infty}\right\}\exp(\phi_n(y))\,dy = O(\exp(-Kn^{a/(a+1)})), \tag{14}$$

where $K>0$ depends only on a. Indeed,

$$\mathrm{Re}\,\phi_n(y) = (1-u)^{-a}\left(\frac{\cos(a\cdot\arctan\{-y/(1-u)\})}{(1+y^2/(1-u)^2)^{a/2}} - 1 - \frac{au}{2(1-u)}\log(1+y^2/u^2)\right),$$

so that

$$\begin{aligned}\int_{\frac{1}{2}(1-u)}^{u} |\exp\phi_n(y)|\,dy &= \int_{\frac{1}{2}(1-u)}^{u} \exp(\mathrm{Re}\,\phi_n(y))dy \\ &\leqq \int_{\frac{1}{2}(1-u)}^{u} \exp\{(1-u)^{-a}([1+y^2/(1-u)^2)^{-a/2}-1]\}\,dy \\ &< \exp(-\{1-(4/5)^{a/2}\}(1-u)^{-a}),\end{aligned}$$

while

$$\int_u^\infty \exp(\operatorname{Re}\phi_n(y))dy \leqq \int_u^\infty \exp\{-\tfrac{1}{2}(n+1)\log(1+y^2/u^2)\}dy$$

$$< \int_u^\infty \exp\{-\tfrac{1}{2}(n+1)\log(2y/u)\}dy$$

$$= \frac{2u^{-\frac{1}{2}(n+1)}}{n+1} < 2\exp(-\tfrac{1}{2}n\log 2).$$

Since $\operatorname{Re}\phi_n(y)$ is even, the proof of (14) is complete.

Now we concentrate on $[-\frac{1}{2}(1-u), \frac{1}{2}(1-u)]$. Setting $y = (1-u)x/2$, we have

$$\int_{-\frac{1}{2}(1-u)}^{\frac{1}{2}(1-u)} \exp\phi_n(y)dy = \tfrac{1}{2}(1-u)\int_{-1}^{1} \exp\{(1-u)^{-a}h_n(x)\}dx, \tag{15}$$

where

$$h_n(x) = \exp\{-a\log(1-ix/2)\} - 1 - \frac{au}{1-u}\log\left(1+\frac{i(1-u)x}{2u}\right).$$

Setting $x = (1-u)^{a/2}v$, we have

$$\tfrac{1}{2}(1-u)\int_{-1}^{1} \exp\{(1-u)^{-a}h_n(x)\}dx = \tfrac{1}{2}(1-u)^{(a+2)/2}\int_{-\infty}^{\infty} g_n(v)dv, \tag{16}$$

where $g_n(v)$ equals $\exp\{(1-u)^{-a}h_n([1-u]^{a/2}v)\}$ when $|v| \leqq (1-u)^{-a/2}$, and zero otherwise.

We now claim that

$$\lim_{n\to\infty} g_n(v) = \exp(-8^{-1}a(a+1)v^2) \qquad (-\infty < v < \infty), \tag{17}$$

and

$$\exists\,\mu > 0 \quad \text{such that} \quad |g_n(v)| \leqq e^{-\mu v^2} \qquad (n = 1,2,3,\cdots). \tag{18}$$

By dominated convergence, (17) and (18) yield

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} g_n(v)dv = \int_{-\infty}^{\infty}\exp(-8^{-1}a(a+1)v^2)dv = 2\left(\frac{2\pi}{a(a+1)}\right)^{\frac{1}{2}}. \tag{19}$$

By combining (12), (13), (14), (15), (16), and (19), Lemma 2 is proved. Thus, it remains to prove (17) and (18).

Since $h_n(x) = -8^{-1}\{a(a+1) + a(1-u)/u\}x^2 + O(x^3)$ uniformly in n, it follows that, for any v,

$$g_n(v) = \exp(-8^{-1}\{a(a+1) + a(1-u)/u\}v^2) + O((1-u)^{a/2}v^3)$$

for sufficiently large n, which implies (17).

To prove (18), note that

$$\operatorname{Re} h_n(x) < \frac{\cos(a \cdot \arctan(x/2))}{(1+x^2/4)^{a/2}} - 1 < 0 \qquad (x \neq 0).$$

Now let

$$k(x) = \begin{cases} \dfrac{1}{x^2}\left(\dfrac{\cos(a \cdot \arctan(x/2))}{(1+x^2/4)^{a/2}} - 1\right) & (x \neq 0) \\ -8^{-1}a(a+1) & (x = 0). \end{cases}$$

$k(x)$ is continuous and negative in $[-1, 1]$, and therefore attains its (negative) maximum there, say $-\mu$. Thus, for all n,

$$\operatorname{Re} x^{-2}h_n(x) \leqq k(x) \leqq -\mu \qquad (-1 \leqq x \leqq 1).$$

This implies that $|g_n(v)| \leqq e^{-\mu v^2}$ for $|v| \leqq (1-u)^{-a/2}$. Since $g_n(v) = 0$ for $|v| > (1-u)^{-a/2}$, (18) is proved.

Now we are in a position to prove

THEOREM 3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in the unit disc. If $|a_n| = \exp(O(n^{1/(p+1)-\varepsilon}))$ for some $\varepsilon > 0$, $(0 < p < \infty)$, then $f \in B^p$.*

PROOF. First let us note $1/(p+1) - \varepsilon < c/(c+1)$ for some $c < 1/p$. Now choose a satisfying $c < a < 1/p$. It is clear that

$$\begin{aligned} |a_n|^2 &= \exp(O(n^{c/(c+1)})) \\ &= O(n^{-(a+2)/(2a+2)} \exp\{(a^{-a/(a+1)} + a^{1/(a+1)})n^{a/(a+1)}\}). \end{aligned} \tag{20}$$

We want to apply Jensen's inequality $\phi(\int g d\theta) \leqq \int \phi(g) d\theta$ with $\phi(t) = \exp(2t^{1/p})$, and $g(\theta) = \log^p |f(re^{i\theta})|$. Noting that $\phi''(t) > 0$ for $t > ((p-1)/2)^p$ when $p > 1$, while for $0 < p \leqq 1$, $\phi''(t)$ is positive for all positive t, we set

$$E_r = \begin{cases} \{\theta \mid \ |f(re^{i\theta})| \geqq e^{\frac{1}{2}(p-1)}\} & (p > 1) \\ \{\theta \mid \ |f(re^{i\theta})| \geqq 1\} & (0 < p \leqq 1). \end{cases} \tag{21}$$

Now Jensen's inequality yields, for $0 < r < 1$,

$$\exp\left(2\left\{\frac{1}{\mu(E_r)}\int_{E_r}\log^p|f(re^{i\theta})|\,d\theta\right\}^{1/p}\right)$$

$$\leqq \frac{1}{\mu(E_r)}\int_{E_r}|f(re^{i\theta})|^2 d\theta \leqq \frac{2\pi}{\mu(E_r)}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2 d\theta$$

$$= \frac{2\pi}{\mu(E_r)}\sum_{n=0}^{\infty}|a_n|^2 r^{2n}.$$

Combining this with (20) and Lemma 2, we have

$$\exp\left(2\left\{\frac{1}{\mu(E_r)}\int_{E_r}\log^p|f(re^{i\theta})|\,d\theta\right\}^{1/p}\right) < \frac{K}{\mu(E_r)}\exp((1-r^2)^{-a}), \tag{22}$$

for some $a < 1/p$, where K is a constant independent of r. Now let A_p equal $((p-1)/2)^p$ for $p > 1$, and zero for $p \leqq 1$. From (21) and (22) it follows that

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}(\log^+|f(re^{i\theta})|)^p d\theta\\ &\leqq A_p + \frac{\mu(E_r)}{2\pi}\frac{1}{\mu(E_r)}\int_{E_r}\log^p|f(re^{i\theta})|\,d\theta\\ &< A_p + \left\{\frac{1}{2}\left(\frac{\mu(E_r)}{2\pi}\right)^{1/p}\cdot\log\frac{K}{\mu(E_r)} + \frac{1}{2}(1-r^2)^{-a}\right\}^p.\end{aligned} \tag{23}$$

Since $\{\mu(E_r)\}^{1/p}\log 1/\mu(E_r)$ is bounded as $r \to 1^-$, the right hand side of (23) is integrable with respect to r on $[0,1]$ whenever $(1-r^2)^{-ap}$ is, i.e., whenever $a < 1/p$, which is indeed the case. Thus

$$\iint_{|z|<1}(\log^+|f(z)|)^p dxdy = \int_0^1\int_0^{2\pi}(\log^+|f(re^{i\theta})|)^p d\theta r dr < \infty. \quad \blacksquare$$

We are now in a position to prove the following

PROPOSITION. (i) *For every $\varepsilon > 0$ and $M > 0$, there exists an $f \in B^p$ with ordered zero set $\{z_n\}$ such that, for sufficiently large n,*

$$1 - |z_n| \geqq Mn^{-(p/(p+1)+\varepsilon)} \qquad (0 < p < \infty).$$

(ii) *For every $\delta > 0$ there exists an $f \in B^p$ such that*

$$\sum_{n=0}^{\infty}(1-|z_n|)^{1+(1/p)-\delta} = \infty \qquad (0 < p < \infty).$$

Note that (i) says that the exponent of n in Theorem 2 is sharp, while (ii) shows that the $1 + 1/p$ in the exponent of $1 - |z_n|$ in the Corollary is the best possible.

(It is not yet clear whether one may dispense with the additive ε in the exponent of the Corollary.)

However, in spite of the fact that Theorem 2 gives a necessary condition in terms of the moduli of the zeros, which is essentially the best possible, one cannot hope that this is anywhere near a sufficient condition for inclusion in B^p. Indeed, since $(\log^+|f(z)|)^p$ is subharmonic for $p \geqq 1$, the fact that f belongs to B^p implies (in a manner similar to Inequality (1)), that

$$|f(z)| \leqq \exp \frac{A}{(1-|z|)^{2/p}}.$$

In particular, let $p > 2$. It now follows from a theorem in Shapiro and Shields [4] that those zeros of f which lie on a single radius satisfy the Blaschke condition $\Sigma(1-|z_n|) < \infty$.

Proof of Proposition. Given $\varepsilon > 0$, let $\beta = 1/(p+1) - \varepsilon$. As in the proof of Theorem 1, let

$$f(z) = \prod_{k=1}^{\infty} \left(1 + b_k z^{[n_k - n_{k-1}]}\right),$$

but this time set $n_k = 2^k$, and $b_k = \exp(M n_k^\beta)$, (where M is an arbitrary positive number). Now for $n_{k-1} < n \leqq n_k$, we have

$$1 - |z_n| = 1 - \exp\left(\frac{-M 2^{\beta k}}{2^{k-1}}\right) > M 2^{(\beta-1)(k-1)} = M n_{k-1}^{-(1-\beta)}$$

$$> M n^{-(1-\beta)} = M n^{-(p/(p+1)+\varepsilon)},$$

where the first inequality holds for sufficiently large k. Thus, it remains to be shown that $f \in B^p$.

As before, let $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$. For $m = 1, 2, 3, \cdots$, we have

$$a_{n_m} = \prod_{k=1}^{m} b_k = \exp\left(M \sum_{k=1}^{m} 2^{\beta k}\right) < \exp(K n_m^\beta),$$

where $K = 2^{\beta+1} M/(2^\beta - 1)$. Clearly, $a_n \geqq 0$ for all n. Also, if $n < n_m$, then $a_n < a_{n_m}$. Thus, for $n_{m-1} < n \leqq n_m$, we have

$$a_n \leqq a_{n_m} < \exp(K n_m^\beta) = \exp(2^\beta K n_{m-1}^\beta) < \exp(2^\beta M n^\beta).$$

By Theorem 3, f belongs to B^p, and (i) is proved. Part (ii) follows trivially.

Theorem 4. *Let $q \geqq 1$. For any $p < q$, there exists a B^p zero set which is not a B^q zero set.*

PROOF. Choose $\varepsilon > 0$ satisfying $p/(p+1) + \varepsilon < q/(q+1)$. By the Proposition, there exists an $f \in B^p$ whose ordered zero set satisfies $1 - |z_n| \geqq n^{-(p/(p+1)+\varepsilon)}$ for sufficiently large n. By Theorem 2, this cannot be the zero set of a B^q function.

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REFERENCES

1. B. Epstein, *On the zeros of analytic A^p functions in the unit disc* (mimeographed).
2. C. Horowitz, Ph.D. thesis, University of Michigan, 1974.
3. C. Horowitz, *Zeros of functions in the Bergman spaces*, Duke Math. J. **41** (1974), 693–710.
4. H. S. Shapiro and A. L. Shields, *On the zeros of functions with finite Dirichlet integral and some related function spaces*, Math. Z. **80** (1962), 217–229.

DEPARTMENT OF MATHEMATICS
BAR-ILAN UNIVERSITY
RAMAT GAN, ISRAEL